# Math 565: Functional Analysis
# Lecture 21

**Cor.** For a Hilbert space $H$, $H^*$ is a Hilbert space, i.e. its (operator) norm comes from an inner product, namely: $\langle f, g\rangle_* := \langle \iota^{-1}(g), \iota^{-1}(f)\rangle$, i.e. $\langle f_x, f_y\rangle_* := \langle y, x\rangle$, where $\iota : H \xrightarrow{\sim} H^*$ is $x \mapsto f_x$.

**Proof.** We only need to check that $\langle \alpha f_x, f_y\rangle_* = \alpha \langle f_x, f_y\rangle_*$:

$$\langle \alpha f_x, f_y\rangle_* = \langle f_{\bar\alpha x}, f_y\rangle_* = \langle y, \bar\alpha x\rangle = \alpha\langle y, x\rangle = \alpha\langle f_x, f_y\rangle_*.$$

We also check $\|f_x\|^2 = \langle f_x, f_x\rangle_*$: $\|f_x\|^2 = \|x\|^2 = \langle x, x\rangle = \langle f_x, f_x\rangle_*$. $\square$

**Cor.** Let $\iota_H : H \hookrightarrow H^* : x \mapsto f_x$ and $\iota_{H^*} : H^* \to H^{**} : f \mapsto F_f$ be the natural maps as in Riesz rep. Then $\iota_{H^*} \circ \iota_H$ is the map $x \mapsto \hat{x} : H \to H^{**}$. In particular, Hilbert spaces are reflexive.

**Proof.** Fix $x \in X$. Then $\iota_H(x) = f_x$ and $F_{f_x}(f_y) := \langle f_y, f_x\rangle_* = \langle x, y\rangle = f_y(x) = \hat{x}(f_y)$. $\square$

## Adjoints.

Let $T \in B(H) := B(H,H)$. Then for each $y \in H$, the map $x \mapsto \langle Tx, y\rangle : H \to \mathbb{C}$ is a bdd linear functional (by CBS, $|\langle Tx, y\rangle| \le \|Tx\|\|y\| \le \|T\|\|y\|\|x\|$ so the norm of this map is $\le \|T\|\|y\|$). Indeed, $\langle Tx, y\rangle = f_y(Tx) = f_y \circ T(x)$, so it the bdd lin. func. $f_y \circ T$. By Riesz, $f_y \circ T = f_z$ for some unique $z \in H$, i.e. $\langle Tx, y\rangle = \langle x, z\rangle$ for all $x \in H$. We denote $T^* y := z$. The map $T^* : H \to H$ is called the **adjoint** of $T$ and it is unique determined by property:

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \quad \text{for all } x, y \in H. \qquad (*)$$

**Prop.** Let $H$ be a Hilbert space and $T \in B(H)$.

$$\begin{tikzcd} H^* & H^* \arrow[l, "T^t"'] \\ H \arrow[u, "\iota"] & H \arrow[u, "\iota"'] \arrow[l, "T^*"] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \quad \text{where } T : H \to H.$$

(a) $T^* \in B(H)$, in fact, $\|T^*\| = \|T\|$.

(b) $T^* = \iota^{-1} \circ T^t \circ \iota$, where $\iota : H \xrightarrow{\sim} H^* : x \mapsto f_x$.

(c) $T^{**} = T$ and $(S \circ T)^* = T^* \circ S^*$.

(d) $(\operatorname{im} T)^\perp = \ker T^*$ and $(\ker T)^\perp = \overline{\operatorname{im} T^*}$.

Proof. (a) To check $T^*(\alpha y+\beta z) = \alpha T^*y + \beta T^*z$, we just need to show that $\alpha T^*y + \beta T^*z$ satisfies (*): for each $x \in H$,

$$\langle x, \alpha T^*y + \beta T^*z\rangle = \bar{\alpha}\langle x, T^*y\rangle + \bar{\beta}\langle x, T^*z\rangle = \bar{\alpha}\langle Tx, y\rangle + \bar{\beta}\langle Tx, z\rangle = \langle Tx, \alpha y + \beta z\rangle$$
$$= \langle x, T^*(\alpha y+\beta z)\rangle.$$

To verify $\|T^*\| = \|T\|$, we use the expression of norm via max inner product:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1}\sup_{\|y\|=1} |\langle Tx, y\rangle| = \sup_{\|y\|=1}\sup_{\|x\|=1} |\langle x, T^*y\rangle| = \sup_{\|y\|=1} \|T^*y\| = \|T^*\|.$$

(b) Check $T^t \circ \iota = \iota \circ T^*$: $T^t \circ \iota(y) = T^t(f_y) = f_y \circ T = f_{T^*y} = \iota(T^*y) = \iota \circ T^*(y)$.

(c) Check (*).

(d) <u>Obs</u> (0 via inner product). For each $x \in H$, $x = 0 \iff \langle x, y\rangle = 0 \ \forall y \in H$. ($\Leftarrow$. Take $y := x$.)

Fix $y \in H$. Then $y \in \ker T^* \iff T^*y = 0 \overset{\text{Obs.}}{\iff} \langle x, T^*y\rangle = 0$ for all $x \in H$

$\iff \langle Tx, y\rangle = 0$ for all $x \in H \iff y \in (\operatorname{im} T)^\perp$.

This proves $\ker T^* = (\operatorname{im} T)^\perp$. For $(\ker T)^\perp = \overline{\operatorname{im} T^*}$, apply the first part to $T^*$:

$(\overline{\operatorname{im} T^*})^\perp = (\operatorname{im} T^*)^\perp = \ker T^{**} = \ker T$, so $(\ker T)^\perp = (\overline{\operatorname{im} T^*})^{\perp\perp} = \overline{\operatorname{im} T^*}$. □

<u>Algebraic characterization of projection.</u> Let $H$ be Hilbert space. An operator $P \in B(H)$ is an orthogonal projection (onto ~~a closed~~ a closed subspace, namely, $P(H)$) $\iff P^2 = P = P^*$.

Proof. $\Rightarrow$. Fix $x \in H$. Then $x = P(x) + (x - P(x)) = x_0 + x_1$, where $x_0 \in P(H)$ and $x_1 \in P(H)^\perp$.

Clearly $P^2(x) = P(x)$ and $\forall x, y \in H$, $\langle Px, y\rangle = \langle x_0, y_0 + y_1\rangle = \langle x_0, y_0\rangle = \langle x_0 + x_1, y_0\rangle = \langle x, Py\rangle$,

so $P = P^*$.

$\Leftarrow$. $P^2 = P$ gives $\forall x \in H$, $x - P(x) \in \ker P$ because $P(x - P(x)) = P(x) - P^2(x) = P(x) - P(x) = 0$.

Also, $(\overline{\operatorname{im} P})^\perp = (\operatorname{im} P)^\perp = \ker P^* = \ker P$. Now if $x \in \overline{\operatorname{im} P}$ then $x - P(x) \in \ker P \perp \overline{\operatorname{im} P}$, but on the other hand $x - P(x) \in \overline{\operatorname{im} P}$ hence $x - P(x) = 0$ so $x = P(x) \in \operatorname{im} P$, i.e. $\operatorname{im} P$ is closed. Furthermore, $x - P(x) \in \ker P$ hence $x - Px \perp \operatorname{im} P$, so $P = \operatorname{proj}_{\operatorname{im} P}$. □

Recall that for Hilbert spaces $H_1, H_2$, a map $U : H_1 \to H_2$ is called unitary if $U$ is a bijection and preserves the inner product: $\langle Uh, Uh'\rangle = \langle h, h'\rangle$ for all $h, h' \in H_1$. By polarization identity, every isometric linear isomorphic is unitary.

Algebraic characterization of unitaries. Let $H$ be a Hilbert space. An operator $U \in B(H)$ is unitary $\iff$ $U$ is a bijection and $U^* = U^{-1}$.

Proof. $\Rightarrow$. We show that $U^{-1}$ fulfills the role of $U^*$: for all $x, y \in H$,

$$\langle x, U^{-1}y\rangle = \langle Ux, UU^{-1}y\rangle = \langle Ux, y\rangle.$$

$\Leftarrow$. Only need to show that $U$ preserves the inner product: for all $x, y \in H$,

$$\langle Ux, Uy\rangle = \langle x, U^*Uy\rangle = \langle x, U^{-1}Uy\rangle = \langle x, y\rangle. \quad \square$$

## Orthonormal bases.

Let $H$ be an inner product space. A family $\{u_i\}_{i\in I}$ is called orthonormal if $\|u_i\| = 1$ and $\langle u_i, u_j\rangle = 0$ $\forall i \neq j$ in $I$.

Prop. For any finite orthonormal family $\{u_i\}_{i\in I}$, $I$ finite, and all $x \in H$,

$$\mathrm{proj}_M x = \sum_{i\in I} \mathrm{proj}_{u_i} x = \sum_{i\in I} \langle x, u_i\rangle u_i, \text{ where } M := \mathrm{span}\{u_i\}_{i\in I}. \text{ In other words,}$$

$$x - \sum_{i\in I} \langle x, u_i\rangle u_i \perp \mathrm{span}\{u_i\}_{i\in I}.$$

Proof. We just verify the last statement: $\langle x - \sum_{i\in I} \langle x, u_i\rangle u_i, u_j\rangle = \langle x, u_j\rangle - \langle x, u_j\rangle = 0$. $\square$

Cor (Bessel's inequality). Let $\{u_i\}_{i\in I}$ be an orthonormal family in an inner prod. space $H$, where $I$ is arbitrary. Then for all $x \in H$,

$$\sum_{i\in I} |\langle x, u_i\rangle|^2 \leq \|x\|^2.$$

In particular, $I_x := \{i \in I : \langle x, u_i\rangle \neq 0\}$ is ctbl.

Proof. It is enough to show that for any finite $I_0 \subseteq I$, $\sum_{i\in I_0} |\langle x, u_i\rangle|^2 \leq \|x\|^2$. (HW)
But this follows from the previous prop. by Pythagoras: letting $M_0 := \mathrm{span}\{u_i\}_{i\in I_0}$,
$\|\mathrm{proj}_M x\|^2 \leq \|x\|^2$ and $\|\mathrm{proj}_M x\|^2 = \|\sum_{i\in I_0} \langle x, u_i\rangle u_i\|^2 = \sum_{i\in I_0} |\langle x, u_i\rangle|^2$. $\square$

**Def.** Let $X$ be a normed vector space and $\{x_i\}_{i\in I} \subseteq X$ be a (potentially unctbl) family of vectors. We say that the series $\sum_{i\in I} x_i$ converges (in norm) to a vector $x \in X$, and write $\sum_{i\in I} x_i = x$, if all but ctbly many $x_i$ are $0$, i.e. $I_0 := \{i\in I : x_i \neq 0\}$ is ctbl, and for any bijective enumeration $I_0 = (i_n)_{n<N}$, where $N \in \mathbb{N}\cup\{\infty\}$, the series $\sum_{n<N} x_{i_n}$ converges to $x$ in norm.

**Def.** Let $H$ be an inner product space. A family $\{u_i\}_{i\in I} \subseteq H$ is called an orthonormal basis (for $H$) if it is orthonormal and $x = \sum_{i\in I} \langle x, u_i\rangle u_i$ for each $x\in H$. In particular, $\langle x, u_i\rangle = 0$ for all but ctbl $i \in I$.

**Prop** (ON basis $\neq$ Hamel basis). If $H$ is an $\infty$-dim Hilbert space, then no orthonormal basis $\{u_i\}_{i\in I}$ is a linear (Hamel) basis.

**Proof.** Because $\dim(H) = \infty$, $I$ has to be infinite so it contains a ctbly infinite subset, WLOG, $\mathbb{N} \subseteq I$. Let $M = \operatorname{span}\{u_n\}_{n\in\mathbb{N}}$. By Baire category, $\{u_n\}_{n\in\mathbb{N}}$ is not a linear (Hamel) basis for $\overline{M}$, so $\exists\, x \in \overline{M}\setminus M$. We show that $x \notin \operatorname{span}\{u_i\}_{i\in I}$. Indeed, otherwise $x = \sum_{n<N} \alpha_n u_n + \sum_{i\in I_0} \alpha_i u_i$ for some $N\in\mathbb{N}$ and finite $I_0 \subseteq I\setminus\mathbb{N}$. But for each $i \in I\setminus\mathbb{N}$, $u_i \perp \overline{M}$ hence $u_i \perp x$, so $I_0$ must be empty, hence $x\in M$, a contradiction. $\square$